

An Algebraic Classification of Some Solvable Groups of Homeomorphisms

Abstract

We produce two separate algebraic descriptions of the isomorphism classes of the solvable subgroups of the group $PL_o(I)$ of piecewise-linear orientation-preserving homeomorphisms of the unit interval under the operation of composition, and also of the generalized R. Thompson groups F_n . The first description is as a set of isomorphism classes of groups which is closed under three algebraic operations, and the second is as the set of isomorphism classes of subgroups of a countable collection of easily described groups. We show the two descriptions are equivalent.

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1 Introduction

We use the main geometric result of [1] to produce two algebraic descriptions of the solvable subgroups of $PL_o(I)$, the group of orientation-preserving piecewise-linear homeomorphisms of the unit interval with finitely many breaks in slope under the operation of composition. Our results apply equally to any group in the family $\{F_n\}$ of generalized Thompson groups, each of which has a definition as a particular subgroup of $PL_o(I)$. (The groups F_n were introduced by Brown in [7], where they were denoted $F_{n,\infty}$. These groups were later extensively studied by Stein in [22], by Brin and Guzmán in [4] and by Burillo, Cleary, and Stein in [8].)

The first description that we find for the isomorphism classes of the solvable subgroups of $PL_o(I)$ is as the smallest non-empty class \mathcal{R} of isomorphism classes of groups which is closed under three operations,

1. taking subgroups,
2. forming standard restricted wreath products with \mathbf{Z} (i.e., $K \mapsto K \wr \mathbf{Z}$), and
3. forming bounded direct sums (countable direct sums of groups in the class with a universal bound on their derived length).

(Note that we give these operations as operations on groups, not on isomorphism classes of groups. We will generally not track the distinction between a group and its isomorphism class in discussion, in order to simplify our language. An example of this is that we may write that $\mathcal{M} \subset \mathcal{R}$, where \mathcal{M} is a set of groups, all of whose isomorphism classes are elements of \mathcal{R} . If H is a group, we may also write that $H \in \mathcal{R}$ when in fact the isomorphism class of H is an element of \mathcal{R} . This practice should cause the reader no confusion, and will not effect our results. The statements of results in this section will not follow this convention and are formally correct.)

Theorem 1.1 *H is the isomorphism class of a solvable subgroup of $PL_o(I)$ if and only if $H \in \mathcal{R}$.*

For each natural number n , define a group G_n according to the process below. First, define $G_0 = 1$, the trivial group. Now inductively define

$$G_n = \bigoplus_{k \in \mathbf{Z}} (G_{n-1} \wr \mathbf{Z}),$$

for all positive integers n . Let $\mathcal{M} = \{G_n \mid n \in \mathbf{N}\}$. Our second description is as indicated by the theorem below.

Theorem 1.2 *$H \in \mathcal{R}$ if and only if H is the isomorphism class of a subgroup of a group in \mathcal{M} .*

In particular, G is a solvable subgroup of $PL_o(I)$ if and only if G is isomorphic to a subgroup of a group in \mathcal{M} . The proof of the above theorem will depend on the following lemma.

Lemma 1.3 *If G is a solvable subgroup of $PL_o(I)$ with derived length n , then G is isomorphic to a subgroup of G_n .*

It is immediate from construction that the groups in \mathcal{M} are all countable, so the last lemma also has the following corollary.

Corollary 1.4 *If H is a solvable subgroup of $PL_o(I)$, then H is countable.*

Finally, each group F_n contains isomorphic copies of each of the groups in \mathcal{M} , so that we have the following consequence to the above results.

Corollary 1.5 *Let n be any integer with $n \geq 2$, and let H be an isomorphism class of groups. $H \in \mathcal{R}$ if and only if H is the isomorphism class of a solvable subgroup of F_n .*

In [18], [19], Navas uses dynamics to analyze the solvable subgroups of a group of homeomorphisms. His work there is focussed on the group $Diff_+^2(\mathbf{R})$. In [19], he applies his results to show that a finitely generated solvable subgroup of $PL_o(I)$ with connected support is isomorphic to a semi-direct product of a group H with the integers \mathbf{Z} , where H is a group in a particular class of groups. This result is contained in our investigations below, although his techniques are quite different from our own.

In the next paragraph, I give a partial list of other related work. This list spans from work directly related to the group $PL_o(I)$ to less directly related work in the theory of groups acting on one dimensional manifolds and the theory of the Godbillon-Vey invariant.

We begin by mentioning the work of Brin and Squier, and later of Brin, on piecewise linear groups of homeomorphisms on the line and the unit interval, [5, 6, 2, 3]. We will refer to this work throughout the paper. Also directly impacting the theory of $PL_o(I)$ are the works of Tsuboi, Minakawa, and Oikonomides. In [23], Tsuboi investigates aspects of $PL_o(I)$ while pursuing his analysis of the Godbillon-Vey invariant, and in [24], he gives another description of the Higman-Thompson group T via restricted actions of $SL(2; \mathbf{Z})$ on the unit circle. Minakawa in [16, 17] provides an invariant that classifies centralizers in $PL_o(I)$, using different techniques than those of by Brin and Squier in [6]. Oikonomides in [21] also analyzes the Godbillon-Vey invariant, and her dissertation contains Minakawa's characterization of the centralizers of elements of $PL_o(I)$ as well. In another direction, Burslem and Wilkinson in [9] classify the solvable subgroups of the group of analytic diffeomorphisms of S^1 . In [11, 12, 13], Farb and Franks analyze groups of homeomorphisms of one-manifolds in analogy with the theory of Lie groups and their discrete subgroups. Finally, the reader is referred to the survey [14] by Ghys for more information about groups acting on the circle.

The author would like to thank Matt Brin for asking for an algebraic interpretation of the geometric results of [1], which lead to all of the results in this paper. The author would also like to thank Étienne Ghys, Andres Navas, and Takashi Tsuboi for mentioning various related works.

The results in this paper are contained in the author's dissertation written at Binghamton University, although some of the proofs here are new.

1.1 Key examples

In this section, we will mention some key examples. All of these examples can be realized in $PL_o(I)$, and in any generalized Thompson group F_n . (We will only specifically realize these groups in R. Thompson's group $F = F_2$ and in $PL_o(I)$; realizations of our examples in the other groups F_n are easy to find by making obvious changes in the constructions below.)

Our examples rely on an understanding of the standard restricted wreath product. The reader is referred to the paper by P. M. Neumann [20] or the book by J. D. P. Meldrum [15] for detailed discussions of these products. For instance, the reader who has read the first section of the first chapter of [15] will be well prepared for the discussion in this paper. Nonetheless, a working definition of a standard restricted wreath product of groups is given below.

Let A and T be groups. The restricted wreath product $A \wr T$ can be thought of as the semidirect product $B \rtimes T$ where $B = \bigoplus_{t \in T} A$ and where the copy of T in the semi-direct product acts on B by right multiplication on the index in the direct sum. In this context T is called the top group in the product, A is called the bottom group, and B is called the base group. We may identify the groups B and T with their natural images in the semidirect product definition without comment.

Taking a restricted wreath product with \mathbf{Z} is something that is easy to realize in $PL_o(I)$, which is why this activity plays a key role in the definition of the groups in the class \mathcal{M} , and in the class \mathcal{R} . In fact, it is so natural that another collection of groups becomes relevant. Define $W_0 = 1$, the trivial group, and for all $i \in \mathbf{N}$, define $W_i = W_{i-1} \wr \mathbf{Z}$, so that

$$W_i = (\dots (((\mathbf{Z} \wr \mathbf{Z}) \wr \mathbf{Z}) \wr \mathbf{Z}) \dots) \wr \mathbf{Z},$$

where there are i appearances of \mathbf{Z} on the right. Theorem 1.2 can be rephrased in terms of the W_i instead of the G_i . However, the W_i lack the corresponding first and third properties of the G_i stated below, and these are serious deficiencies in the class, from a computational point of view.

We leave the first property below to the reader. We will prove the latter two later.

Remark 1.6 1. *Given any non-negative $i \in \mathbf{Z}$, $G_i \cong \bigoplus_{j \in \mathbf{Z}} G_i$ (note: the subscript is not the sum index).*

2. *Given non-negative $n \in \mathbf{Z}$, G_n has derived length n .*

3. *If H is a subgroup of G_k for some non-negative $k \in \mathbf{Z}$, and H has derived length n , then H is isomorphic to a subgroup of G_n .*

1.2 Geometry

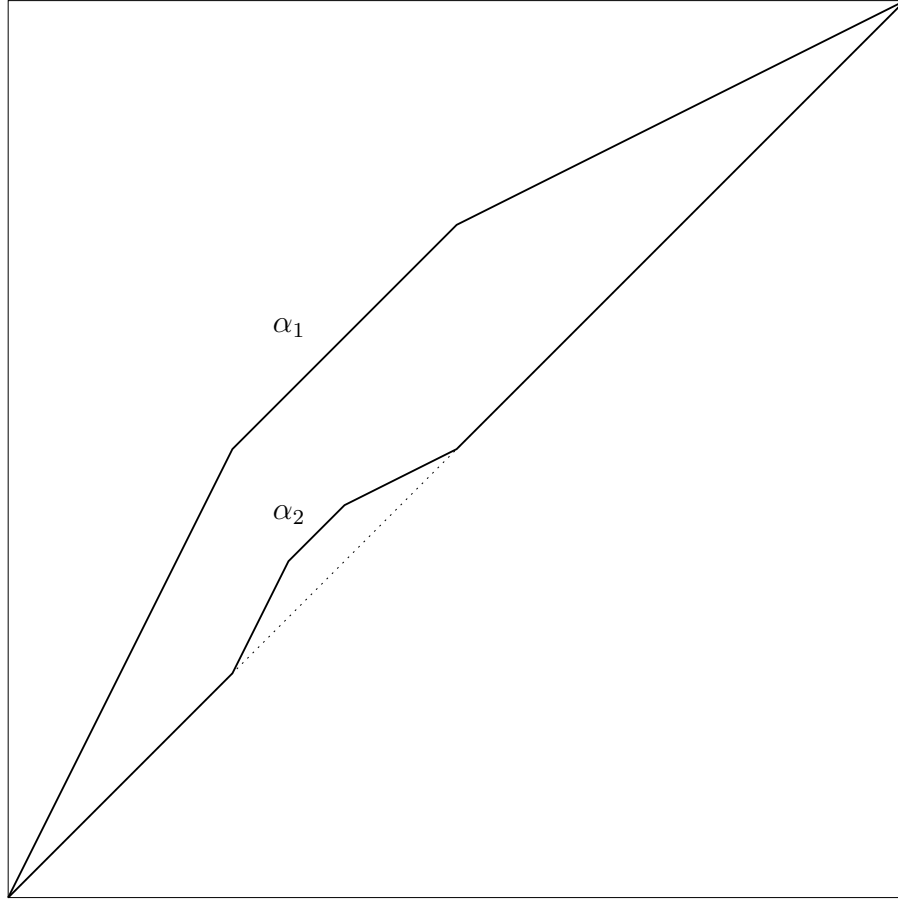
In this section, we will realize the W_i in $F \leq PL_o(I)$, and use these realizations to motivate some geometric definitions which will enable us to state the main geometric result of [1]. Here, we are thinking of F as the realization of Thompson's group in $PL_o(I)$ which consists of all the elements of $PL_o(I)$ which have all slopes powers of two, and which have all breakpoints occurring at the dyadic rationals $\mathbf{Z}[\frac{1}{2}]$. See Cannon, Floyd, and Parry [10] for an introduction to the remarkable group F .

1.2.1 Realizing the W_i

Consider the two elements $\alpha_1, \alpha_2 \in PL_o(I)$ defined below:

$$x\alpha_1 = \begin{cases} 2x & 0 \leq x < \frac{1}{4}, \\ x + \frac{1}{4} & \frac{1}{4} \leq x < \frac{1}{2}, \\ \frac{1}{2}x + \frac{1}{2} & \frac{1}{2} \leq x \leq 1, \end{cases} \quad x\alpha_2 = \begin{cases} x & 0 \leq x < \frac{1}{4}, \\ 2x - \frac{1}{4} & \frac{1}{4} \leq x < \frac{5}{16}, \\ x + \frac{1}{16} & \frac{5}{16} \leq x \leq \frac{3}{8}, \\ \frac{1}{2}x + \frac{1}{4} & \frac{3}{8} \leq x < \frac{1}{2}, \\ x & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Here are the graphs (superimposed) of these functions:



Either element alone generates a group isomorphic to $\mathbf{Z} \cong W_1$ in $PL_o(I)$, but the “action” of α_2 occurs in a single fundamental domain of α_1 ; that is, $\frac{1}{4}\alpha_1 = \frac{1}{2}$, but α_2 is the identity off of the interval $[\frac{1}{4}, \frac{1}{2}]$. In particular, $\alpha_2^{\alpha_1} = \alpha_1^{-1}\alpha_2\alpha_1$ has support $(\frac{1}{2}, \frac{3}{4})$, which is disjoint from the support of α_2 . (In this discussion, following the notation in Brin’s papers [2] and [3], elements of $PL_o(I)$ act on the right on I , and the support of any particular element of

$PL_o(I)$ is the open set of points in I that are moved by that element.) In particular, any two distinct conjugates of α_2 by powers of α_1 commute with each other, since their supports will be disjoint in the interval I . Now, if we consider any element $h \in \langle \alpha_1, \alpha_2 \rangle$, it is a standard algebraic fact that we can write h as a product of the form a power of α_1 followed by a product of conjugates of α_2 (and α_2^{-1}) by various powers of α_1 . In particular, the element α_1 generates a group T isomorphic to \mathbf{Z} which acts on the normal subgroup B of $\langle \alpha_1, \alpha_2 \rangle$ generated by the conjugates of α_2 by different powers of α_1 . Since these conjugates all have disjoint support, the group B is isomorphic to a direct sum of copies of the integers. The following chain of isomorphisms should now make sense:

$$\langle \alpha_1, \alpha_2 \rangle \cong \left(\bigoplus_{i \in \mathbf{Z}} \mathbf{Z} \right) \rtimes \mathbf{Z} \cong \mathbf{Z} \wr \mathbf{Z} \cong W_2.$$

Note before we move on that α_1 and α_2 are both elements of Thompson's group F .

All of the W_i can be realized in Thompson's group F in an entirely similar way, using one-orbital generators; take α_i to be the conjugate $\alpha_i = \alpha_{i-1}^s = s^{-1}\alpha_{i-1}s$, where s is the “shrinking function” $s(x) = \frac{1}{4}x + \frac{1}{4}$, and the conjugation takes place in $PL_o(\mathbf{R})$, where we replace α_1 by the function in $PL_o(\mathbf{R})$ which behaves as the identity outside of the unit interval $[0, 1]$ for this inductive definition. Given $i \in \mathbf{N}$, $i \geq 2$, the support of α_i is contained in a single fundamental domain of α_{i-1} , so that given $j \in \mathbf{N}$, $j \geq 1$, we have $W_j \cong \langle \alpha_1, \alpha_2, \dots, \alpha_j \rangle$. Since conjugating any element of F by s will still produce an element of F , we see that the W_i 's can all be realized in F .

1.2.2 Orbitals, Towers, and Derived Length

Much of the language of this section is motivated by thinking of subgroups of $PL_o(I)$ as permutation groups acting on the set $I = [0, 1]$.

If H is a subgroup of $PL_o(I)$, then its support naturally falls into a collection of disjoint, open intervals. Each such we will call an *orbital* of the group H . Given an element $\gamma \in H$, the group $\langle \gamma \rangle$ has its own orbitals, which are the connected components of the support of γ . We will use the symbol $o\gamma$ to denote the number of orbitals of the group $\langle \gamma \rangle$. Now, given such an interval $A = (a, b) \subset [0, 1]$, we call A an *orbital* of γ , we may also refer to A as an “element orbital.” Since H now must have infinitely many elements that also have A as an orbital, we typically use *signed orbitals*, which are pairs of the form (A, γ) to indicate not only which element orbital we are considering, but also the specific element which is “owning” that orbital in our discussion. Given a signed orbital $s = (A, \gamma)$, we call A the *orbital* of s and we call γ the *signature* of s . Given a set X of signed orbitals, we will use O_X to denote the set of element orbitals that are orbitals of elements of X , and we will use S_X to denote the set of signatures that are signatures of elements of X .

It is a standard fact that $PL_o(I)$ can be totally ordered, and the set of open intervals in I is partially ordered by inclusion, so the set of signed orbitals is a poset under the induced lexicographic order. The ordering on $PL_o(I)$ will not play a role here, except by enabling simplified language in the upcoming definition.

In the example groups W_i , each of the α_k had an orbital whose closure was fully contained in the orbital of α_{k-1} , whenever $k > 1$. The ease with which we can analyze the groups

generated by the elements α_k , which have their orbitals arranged in such a nice “stack”, motivates the following.

Given a group $G \leq PL_o(I)$ and a set T of signed orbitals of G , we will say T is a *tower* of G if T satisfies the following properties:

1. T is a chain in the partial order on the signed orbitals of G .
2. For any orbital $A \in O_T$, T has exactly one element of the form (A, g) .

Let T be a tower of $PL_o(I)$. We will call the cardinality of T its *height*, using the simple descriptive *infinite* if T has an infinite cardinality.

We are now in a position to approach the main geometric result of [1]. We will say that a group has depth $n \in \mathbf{N}$ if and only if we can find towers of height n , but no towers of height $n + 1$, in the group. The main result of [1] is as follows:

Theorem 1.7 *Suppose G is a subgroup of $PL_o(I)$ and $n \in \mathbf{N}$. G is solvable with derived length n if and only if G has depth n .*

This geometric result is the cornerstone upon which the results of this paper are built.

2 Classification of solvable subgroups in $PL_o(I)$

In this section we will pursue the algebraic classification of the solvable subgroups in $PL_o(I)$. One direction of the classification is mostly algebraic, and requires less knowledge of the terminology of $PL_o(I)$. We will engage in that direction first.

2.1 The class \mathcal{R}

Recall that \mathcal{R} represents the smallest non-empty class of isomorphism classes of groups which is closed under the following three operations (again, given as operations on groups).

1. Restricted wreath product with \mathbf{Z} .
2. Bounded direct sum.
3. Taking subgroups.

It turns out that we can begin to understand \mathcal{R} more deeply via a close study of the class $\mathcal{M} = \{G_i \mid i \in \mathbf{N}\}$ defined in the introduction. Let us gather some facts about the groups G_k . (Below, we will prove the second point in Remark 1.6, which corresponds to the fourth point here.)

Lemma 2.1 *1. If F_0, F_1, H_0 , and H_1 are groups, where $F_0 \leq F_1$ and $H_0 \leq H_1$, then $F_0 \wr H_0 \leq F_1 \wr H_1$.*

2. For any m and $n \in \mathbf{N}$, with $m < n$, G_m embeds as a normal subgroup of G_n .

3. For any group G with derived length n , the groups $G \wr \mathbf{Z}$ and $\bigoplus_{i \in \mathbf{Z}} (G \wr \mathbf{Z})$ have derived length $n + 1$.
4. Given any $n \in \mathbf{N}$, G_n has derived length n .

pf: The first point is immediate by examining the following chain of subgroup inclusions, where the inclusions are based on the underlying sets.

$$\begin{aligned} F_0 \wr H_0 &= \{((f_0)_{h \in H_0}, h_0) \mid (f_0)_{h \in H_0} \in \bigoplus_{a \in H_0} F_0, h_0 \in H_0\} \leq \\ F_1 \wr H_0 &= \{((f_1)_{h \in H_0}, h_0) \mid (f_1)_{h \in H_0} \in \bigoplus_{a \in H_0} F_1, h_0 \in H_0\} \leq \\ F_1 \wr H_1 &= \{((f_1)_{h \in H_1}, h_1) \mid (f_1)_{h \in H_1} \in \bigoplus_{a \in H_1} F_1, h_1 \in H_1\} \end{aligned}$$

To see the second point, we will demonstrate an embedding of G_{n-1} into G_n , and thus inductively define an embedding of G_m into G_n , for any non-negative integers $m < n$. Note that there are many copies of G_{n-1} in G_n , but we are particularly interested in the one given in the next paragraph, which is the copy that we use to inductively define our particular copy of G_m in G_n . The normality of this embedded copy of G_m in G_n follows easily from the theory of group actions, which can be checked in section 2.2 below where we realize G_n in $PL_o(I)$. (Let X be the set of left hand endpoints of the components of the support of G_m in I . X is acted upon by G_n , and the kernel of this action is G_m .) A second proof is by noting that the embedded copy of G_{n-1} in G_n that we demonstrate in the paragraph below is characteristic in G_n , which proof can be carried out with the help of the geometric tools established in the proof of Theorem 1.7. We will not use the normality of our embedded copy of G_m in G_n later.

Now let us describe our particular embedded copy of G_{n-1} in G_n . First, identify the base group of $G_{n-1} \wr \mathbf{Z}$ with G_{n-1} using the fact that $\bigoplus_{i \in \mathbf{Z}} G_{n-1} \cong G_{n-1}$, now since the direct sum of the base groups of the $G_{n-1} \wr \mathbf{Z}$ summands in the definition of G_n is also a subgroup of G_n , we see that $\bigoplus_{i \in \mathbf{Z}} G_{n-1}$ is a subgroup of G_n . But now again, this last direct sum is isomorphic with G_{n-1} , so that G_{n-1} (as embedded here as the direct sum of the base groups of the $G_{n-1} \wr \mathbf{Z}$ summands of G_n) is a subgroup of G_n .

For the third point, it follows from Neumann [20] (Theorem 4.1 and Corollary 4.5) that if $G = A \wr B$, then G' is contained in a sum of copies of A , and also that G' surjects onto A (these facts are under the condition that B is abelian). Both facts are easy exercises in our situation with $B = \mathbf{Z}$. This immediately implies the third point.

The fourth point follows directly from the third.

◇

Now let us examine \mathcal{R} for a short time. We give (modulo work to come later) a characterization of \mathcal{R} that completes one direction of Theorem 1.1. The key result (Lemma 2.4 below) will not be used in the rest of the paper.

Since \mathcal{R} is nonempty and closed under taking subgroups, the trivial group 1 is a group in \mathcal{R} . Let \mathcal{G} represent the class of all isomorphism classes of groups. Define $\mathcal{P} : \mathcal{P}(\mathcal{G}) \rightarrow \mathcal{P}(\mathcal{G})$ to be the closure operator that takes a class X of isomorphism classes of groups and computes the smallest class of isomorphism classes of groups which contains X and is closed under the isomorphism class operations induced from the group operations of restricted wreath product with \mathbf{Z} and bounded direct sum. Since each G_i is obtained by applying a finite sequence of bounded direct sums and wreath products with \mathbf{Z} to the trivial group 1, we see

that $\mathcal{M} \subset \{1\}\mathcal{P}$. By the definition of \mathcal{R} , it is immediate that $\{1\}\mathcal{P} \subset \mathcal{R}$. In particular, we have:

$$\mathcal{M} \subset \{1\}\mathcal{P} \subset \mathcal{R}$$

Now let us consider three operators

$$\begin{aligned}\mathcal{S} &: \mathcal{P}(\mathcal{G}) \rightarrow \mathcal{P}(\mathcal{G}), \\ \mathcal{W} &: \mathcal{P}(\mathcal{G}) \rightarrow \mathcal{P}(\mathcal{G}), \\ \mathcal{B} &: \mathcal{P}(\mathcal{G}) \rightarrow \mathcal{P}(\mathcal{G}),\end{aligned}$$

that represent taking closure under the operations of taking subgroups, taking restricted wreath products with \mathbf{Z} , and building bounded direct sums, respectively.

If Γ is the set of all finite length words of the form $(\mathcal{WB})^k$ or $(\mathcal{BW})^k$ where $k \in \mathbf{N}$, and $X \in \mathcal{P}(\mathcal{G})$, then the union $\Upsilon_X = \cup_{\gamma \in \Gamma} X\gamma$ is the smallest closed set containing X that is closed under both operations of taking bounded direct sums and wreath products with \mathbf{Z} , so that $\Upsilon_{\{1\}} = \{1\}\mathcal{P}$.

Lemma 2.2 *Let m be a non-negative integer. If G is a group in $\{1\}\mathcal{P}$ with derived length m , then G is isomorphic to a subgroup of G_m .*

pf: We can prove this by inducting on the derived length of G . If G has derived length 0, then G is the trivial group so $G = G_0$. Let $n \in \mathbf{N}$, and suppose G has derived length n and that for any group H in $\{1\}\mathcal{P}$ which has derived length m where $0 \leq m < n$, we know that $H \leq G_m$. Now, there is a $j \in \mathbf{N}$ so that $G \in \{1\}(\mathcal{BW})^j$, or $G \in \{1\}(\mathcal{WB})^j$. Note that if $G \in \{1\}(\mathcal{WB})^j$ then $G \in \{1\}(\mathcal{BW})^{j+1}$, so we will assume that $G \in \{1\}(\mathcal{BW})^k$ for some minimal non-negative integer k . There are now two cases:

1. $G \in \{1\}(\mathcal{BW})^{k-1}\mathcal{B}$ but G is not in $\{1\}(\mathcal{BW})^{k-1}$.

In this case, The last operation required to build G was a bounded direct sum of groups, all of which groups have derived lengths necessarily less than or equal to n (note here that a finite sequence of bounded direct sums is isomorphic to a bounded direct sum). The summands of this bounded direct sum are all groups in $\{1\}(\mathcal{BW})^{k-1}$. We will argue that each of these groups is actually a subgroup of G_n , and therefore, since a countable or finite direct sum of groups isomorphic to G_n is actually isomorphic to G_n , we will have finished this case.

Let H be a summand of the final bounded direct sum which created G , and assume that the derived length of H is actually n (at least one summand must have this derived length), and that $H \in \{1\}(\mathcal{BW})^{k-1}$. If H is actually in $\{1\}(\mathcal{BW})^{k-2}\mathcal{B}$, then we can replace H inductively by a summand (with derived length n) of the last bounded sum operation used to create H , so that there is a $t \in \mathbf{N}$ so that H is now an element of $\{1\}(\mathcal{BW})^t$, but not an element of $\{1\}(\mathcal{BW})^{t-1}\mathcal{B}$. In particular, H is the result of applying s wreath products with \mathbf{Z} to a group H^* in $\{1\}(\mathcal{BW})^{t-1}\mathcal{B}$ for some $s \in \mathbf{N}$. H^* is therefore a group in $\{1\}\mathcal{P}$ with derived length $n - s$, and therefore H^* is a subgroup of G_{n-s} . But note that for any integer p we have that $G_p \wr \mathbf{Z} \leq G_{p+1}$, so

abusing notation (the parentheses below should accumulate on the left), we see that $H = H^*(\wr \mathbf{Z})^s \leq G_{n-s}(\wr \mathbf{Z})^s \leq G_{n-s+1}(\wr \mathbf{Z})^{s-1} \leq \dots \leq G_n$.

If H is a summand of G with derived length m where $m < n$, Then by our induction hypothesis, $H \leq G_m$, but $G_m \leq G_n$, so $H \leq G_n$.

2. $G \in \{1\}(\mathcal{BW})^k$ but G is not in $\{1\}(\mathcal{BW})^{k-1}\mathcal{B}$.

This case is entirely similar to the last case, except that we already know that G is the result of applying s wreath products with \mathbf{Z} to a group H in $\{1\}(\mathcal{BW})^{k-1}\mathcal{B}$ for some positive integer s . The derived length of H must be $n - s$, and therefore $H \leq G_{n-s}$ so that $G \leq G_n$ as in the penultimate paragraph of the previous case.

◇

We need one last technical lemma before we can complete our exploration of the class \mathcal{R} :

Lemma 2.3 *Suppose m and $n \in \mathbf{N}$, with $m \leq n$. If $H \leq G_n$ and H has derived length m , then H is isomorphic to a subgroup of G_m .*

pf: This follows from Corollary 4.8 below, and the fact that we can realize the groups \mathcal{M} in $PL_o(I)$ (see the next subsection). ◇

Our arguments after this section do not rely on the classification of \mathcal{R} given in the next lemma.

Finally, we have a nice description of \mathcal{R} . Note that the following lemma implies Theorem 1.2.

Lemma 2.4 $\mathcal{R} = \{1\}\mathcal{PS} = \mathcal{MS}$

Pf: We have already shown that $\{1\}\mathcal{P} \subset \mathcal{MS}$, so we know that $\{1\}\mathcal{PS} \subset \mathcal{MS}$, and by definition, $\mathcal{M} \subset \{1\}\mathcal{P}$, so that $\mathcal{MS} \subset \{1\}\mathcal{PS}$. In particular, $\mathcal{MS} = \{1\}\mathcal{PS}$.

We will now show that $\mathcal{MSB} = \mathcal{MS}$ (implying that \mathcal{MS} is already closed under the operation of taking bounded direct sums) and that $\mathcal{MSW} = \mathcal{MS}$ (implying that \mathcal{MS} is already closed under taking wreath products with \mathbf{Z}), so that we can conclude that $\mathcal{MS} = \mathcal{R}$.

To see that $\mathcal{MSB} = \mathcal{MS}$, let $G \in \mathcal{MSB}$. There is an $M \in \mathbf{N}$ so that we can write $G = \bigoplus_{i \in \mathbf{Z}} H_i$, where each H_i has derived length bounded above by M . Now, each $H_i \leq G_M$ (by the Lemma 2.3), so we see that $G \leq \bigoplus_{i \in \mathbf{Z}} G_M$, hence $G \in \mathcal{MS}$.

To see that $\mathcal{MSW} = \mathcal{MS}$, we note that if $G \in \mathcal{MSW}$, then either $G \in \mathcal{MS}$ or we can write $G = ((\dots((H \wr \mathbf{Z}) \wr \mathbf{Z}) \dots) \wr \mathbf{Z})$, where $H \in \mathcal{MS}$, and where there are k wreath products with \mathbf{Z} , for some $k \in \mathbf{N}$. In the first case we are done. In the second case $H \leq G_M$ for some non-negative integer M . But now $G = ((\dots(H \wr \mathbf{Z}) \wr \mathbf{Z}) \dots) \wr \mathbf{Z} \leq ((\dots(G_M \wr \mathbf{Z}) \wr \mathbf{Z}) \dots) \wr \mathbf{Z} = J \in \{1\}\mathcal{P}$, where J has derived length $M + k$, so that $G \leq G_{M+k}$. Finally we have that $G \in \mathcal{MS}$. ◇

2.2 Realizing \mathcal{R} in $PL_o(I)$ and F

Here we will explain how we can realize the groups in \mathcal{M} inside Thompson's group F , as realized in $PL_o(I)$. This will prove one half of Theorem 1.1. To realize the other direction of Theorem 1.1, we will need to take a detour through the geometric definitions of $PL_o(I)$.

The elements $\alpha_1, \alpha_2 \in PL_o(I)$ defined in the introduction will play a major role here, as will the shrinking conjugator s .

First, observe that we can realize G_1 fairly simply. Let $\beta_0 = \alpha_2$, and define β_k for each $k \in \mathbf{Z}$ as $\beta_0^{\alpha_1^k}$. G_1 is immediately isomorphic to $\langle \beta_k | k \in \mathbf{Z} \rangle$, and G_1 has been realized in Thompson's group F .

Now we will show that given any group H which has been realized as a subgroup of $PL_o(I)$, we can realize $\bigoplus_{i \in \mathbf{Z}} (H \wr \mathbf{Z})$ as a subgroup of $PL_o(I)$. Firstly, conjugate the elements of H by s twice, to create a new group H_0 isomorphic with H . The supports of all of the elements of H_0 are contained in the set $(\frac{5}{16}, \frac{3}{8})$, which is contained in a single fundamental domain of $\alpha_2 = \beta_0$. At this juncture, the group generated by H_0 and β_0 is isomorphic to $H \wr \mathbf{Z}$, where β_0 is the generator of the top \mathbf{Z} factor. In particular, we can realize $H \wr \mathbf{Z}$ in $PL_o(I)$. But now $\bigoplus_{i \in \mathbf{Z}} (H \wr \mathbf{Z})$ is the base group of $(H \wr \mathbf{Z}) \wr \mathbf{Z}$, which we can also realize by repeating the previous procedure, so we are done.

Observe that the shrinking map conjugates elements of Thompson's group F into Thompson's group F , so that if H is realized as a subgroup of F , then this construction of $\bigoplus_{i \in \mathbf{Z}} (H \wr \mathbf{Z})$ also produces a subgroup of F . In particular, we see inductively that each group G_n can now be realized in Thompson's group F , and therefore that each group of \mathcal{R} can be realized in Thompson's group F .

3 $PL_o(I)$

We will now build required terminology for working with subgroups of $PL_o(I)$. We will use notation similar to that in Brin's paper [2] on the ubiquitous nature of Thompson's group F in subgroups of $PL_o(I)$. Some portions of the text here were lifted from [1].

We note that the set of points moved by an element h of $PL_o(I)$ is open, by the continuity of elements of $PL_o(I)$. But then the support of H , for any subgroup $H \leq PL_o(I)$, is a countable union of pairwise disjoint open intervals in $(0, 1)$. Let the collection \mathcal{O}_H always denote the countable set of open, pairwise disjoint intervals of the support of H . Recall from section 1.2.2 that we call these intervals the *orbitals* of H . There is a natural total order on \mathcal{O}_H , where if $A, B \in \mathcal{O}_H$, where $A \neq B$, we will say $A < B$ or A is to the left of B if and only if given any $x \in A$ and $y \in B$, we have $x < y$ under the natural order induced by $I \subset \mathbf{R}$. Since A and B are disjoint, connected subsets of \mathbf{R} , this definition does not depend on the choices of x and y .

If the collection \mathcal{O}_H is finite, we may speak of the "first" orbital, or "second" orbital, etc., where the first orbital is the leftmost orbital under the definition given above, the second orbital is the orbital to the left of all other orbitals in \mathcal{O}_H other than itself and the first orbital, and so on.

Given an open interval $A = (a, b) \subset \mathbf{R}$, where $a < b$, we will refer to a as the *leading end* of A , and to b as the *trailing end* of A . If the interval is an orbital of some group $H \in PL_o(I)$, we will refer to the *ends of the orbital* in the same fashion.

If $h \in H$ and $x \in \text{Supp}(h)$, we will say that h moves x to the left if $xh < x$, and we will say that h moves x to the right if $xh > x$. Furthermore, we will say that $x \in I$ is a *breakpoint* for h if the left and right derivatives of h exist at x , but are not equal. We recall that by definition, h will admit only finitely many breakpoints. If \mathcal{B}_h represents the set of

breakpoints of the element h , then $(0, 1) \setminus \mathcal{B}_h$ is a finite collection of open intervals, which we will call *affine components of h* , which admit a natural “left to right” ordering as before. We shall therefore refer to the “first” affine component of h , or the “second” affine component of h , etc. We sometimes will refer to the first affine component of h as the *leading affine component of h* , and to the last affine component of the domain of h as the *trailing affine component of h* .

The following are some useful remarks (whose proofs are left to the reader), that are mostly standard in the theory of $PL_o(I)$.

Remark 3.1 1. If A is an orbital for $h \in H$, then either $xh > x$ for all points x in A , or $xh < x$ for all points x in A .

2. Any element $h \in PL_o(I)$ has only finitely many orbitals.

3. If $h \in PL_o(I)$ and $A = (a, b)$ is an orbital of h , then given any $\epsilon > 0$ and x in A , there is an integer n so that $|xh^{-n} - a| < \epsilon$ and $|xh^n - b| < \epsilon$.

Given an orbital A of H we say that h *realizes an end of A* if some orbital of h lies entirely in A and shares an end with A . Note that Brin uses the word “Approaches” for this concept in [2], but we will use “Approaches” to also indicate the direction in which h moves points, as follows: we will say that h *approaches the end a of A in A* if h has an orbital B where $B \subset A$ and B has end a , and h moves points in B towards a . In particular, h realizes a in A and h moves points in its relevant orbital towards a . If A is an orbital for H then we say that $h \in H$ *realizes A* if A is also an orbital for h .

If g and h are elements of $PL_o(I)$ and there is an interval $A = (a, b) \subset I$ so that both g and h have A as an orbital, then we will say that g and h *share the orbital A* .

3.1 Conjugation and Transition Chains

All statements in this section will be either fairly standard and given without proof, or are proven (or are immediate from proofs) in [1]. First, we will give some of the standard facts.

Let $g, h \in PL_o(I)$ and let $k = g^h = h^{-1}gh$. Suppose that $\mathcal{O}_g = \{A_i\}_{i=1}^n$ are the n orbitals of g in left to right order, where $n \in \mathbf{N}$ and $i \in \{1, 2, \dots, n\}$. Define

$$A_i^* = \{x \in I | xh^{-1} \in A_i\} = A_ih$$

for all $i \in \{1, 2, \dots, n\}$.

Lemma 3.2 $o_k = o_g = n$ and collection $\{A_i^*\}_{i=1}^n$ is the ordered set of orbitals of g^h in left to right order.

In the setting of the above lemma, we will say that the A_ih are the *induced orbitals of k from g by the action of h* . We might also say that the orbitals of k are induced from the orbitals of g by the action of h .

The following is worth pointing out:

Remark 3.3 Suppose $g, h \in PL_o(I)$ and $f = gh$. If b is a breakpoint of f then b is a breakpoint of g or bg is a breakpoint of h .

Finally, our last well known fact.

Lemma 3.4 *If $H \leq PL_o(I)$ and $A = (a, b)$ is an orbital for H , then given any points $c, d \in A$, with $c < d$, there is an element $g \in H$ so that $cg > d$.*

For the rest of the section, we give definitions and basic statements introduced in [1].

The following definition is a special case of a more general definition. In this paper, we will only need objects of the more restricted type. Let $A = (a, b)$ and $B = (c, d)$ be two intervals contained in I , and let $\alpha, \beta \in G$ for some subgroup $G \leq PL_o(I)$ so that the set $\mathcal{C} = \{(A, \alpha), (B, \beta)\}$ is a set of signed orbitals for G . We will say \mathcal{C} is a *transition chain of length two* for G if $a < c < b < d$. The existence of a transition chains of length two in a group allows the creation of complex dynamics in the group action on the unit interval, as one can repeatedly act on the interval with one element to move a point near to an edge of an element orbital, and then use the “next” element to move the point out of its initial element orbital.

3.2 Notes on Towers and Orbitals

In order to finish the proofs of the main theorems of this paper, we will need a deeper understanding of towers. Recall that a tower T is a set of signed orbitals satisfying the following two conditions:

1. T is a chain in the poset of signed orbitals associated with $PL_o(I)$, and
2. for any orbital $A \in O_T$, T has exactly one element of the form (A, g) .

We will refine this definition, following the related section 2.3 in [1].

The notions of depth of groups and height of towers can be extended to other objects.

Given a group $G \leq PL_o(I)$, and an open interval $A \subset I$, we will define the *depth of A in G* to be the supremum of the heights of the finite towers which have their smallest element having the form (A, h) for some element $h \in G$. In particular, if A is not an orbital for an element of G , then the depth of A will be zero. Symmetrically, we define the *height of A in G* to be the supremum of the heights of the finite towers which have their largest element having the form (A, h) for some element $h \in G$.

Towers, as defined, are easy to find, but can be difficult to work with. For an arbitrary tower T , there are no guarantees about how other orbitals of signatures of the elements of T cooperate with the orbitals of the tower. We say a tower T is an *exemplary tower* if the following two additional properties hold:

1. Whenever $(A, g), (B, h) \in T$ then $(A, g) \leq (B, h)$ implies the orbitals of g are disjoint from the set of ends of the orbital B .
2. Whenever $(A, g), (B, h) \in T$ then $(A, g) \leq (B, h)$ implies no orbital of g in B shares an end with B .

3.3 Thompson's group F and balanced subgroups of $PL_o(I)$

Brin showed in [2] the following theorem:

Theorem 3.5 (Ubiquitous F) *If a group $H \leq PL_o(I)$ has an orbital A so that some element $h \in H$ realizes one end of A , but not the other, then H will contain a subgroup isomorphic to Thompson's group F .*

We will say that an orbital A of a group $H \leq PL_o(I)$ is *imbalanced* if some element $h \in H$ realizes one end of A , but not the other, and we will say A is *balanced* if whenever an element $h \in H$ realizes one end of A , then h also realizes the other end of A (note that h might do this with two distinct orbitals). Extrapolating, given a group $H \leq PL_o(I)$, we will say that H is *balanced* if given any subgroup $G \leq H$, and any orbital A of G , every element of G which realizes one end of A also realizes the other end of A . Informally, H has no subgroup G which has an orbital that is “heavy” on one side. In the case where H has a subgroup G with an imbalanced orbital, then we will say that H is *imbalanced*.

Remark 3.6 *If $H \leq PL_o(I)$ and H is imbalanced, then H has a subgroup isomorphic to Thompson's group F .*

Since F' is a non-abelian simple group ([10], Theorem 4.3), F is not solvable. Thus imbalanced groups are not solvable.

The dynamics of balanced groups are much easier to understand than those of imbalanced groups. One indicator of this is that imbalanced groups admit transition chains of length two, and groups which admit transition chains of length two admit infinite towers.

The following lemma sums up what we have learned, from a utilitarian perspective. It is an easy consequence of the results in [1].

Lemma 3.7 *If G is a solvable subgroup of $PL_o(I)$ then*

1. G is balanced,
2. G does not admit transition chains of length two, and
3. all towers for G are exemplary.

3.3.1 A useful homomorphism

The homomorphism in this section was known to Brin and Squier during the research that led to the paper [6].

Let us suppose that $H \leq PL_o(I)$ and $A = (a, b)$ is an orbital of H . To simplify the arguments for now, let us suppose that A is the only orbital of H . We can define a map $\phi : H \rightarrow \mathbf{R} \times \mathbf{R}$ defined by $h \mapsto (h_a, h_b)$ where $h_a = \ln(h'_+(x))$ and $h_b = \ln(h'_-(x))$. Ie., we take the logs of the slopes of h at the ends of A . Since h is a piecewise-linear orientation-preserving homeomorphism of I , we see that the derivatives exist and are positive, and so ϕ is well defined for all $h \in H$. If h does not realize a (resp. b) then we see that h behaves as the identity near a (b) in A , and so $h_a = \ln(1) = 0$ ($h_b = 0$). If $h, g \in H$ then $hg\phi = h\phi + g\phi$ in $\mathbf{R} \times \mathbf{R}$ by the chain rule. In particular, we see the following remark:

Remark 3.8 ϕ is a homomorphism of groups.

Now the image of ϕ is quite interesting, it carries a small amount of the complexity of H , but still enough to allow us to find out if A is an imbalanced orbital. The following straightforward lemma is left to the reader.

Lemma 3.9 *The orbital A is imbalanced if and only if $\text{Im}(\phi)$ contains an element of the form $(\alpha, 0)$ or $(0, \alpha)$ where $\alpha \neq 0$.*

This next technical lemma will help with the lemma that follows it:

Technical Lemma 3.10 *Suppose $H \leq \text{PL}_o(I)$, H has an orbital $A = (a, b)$, and that H has a sequence of elements $(g_n)_{n=0}^\infty$ in H which satisfies the properties below.*

1. *For each $i \in \mathbf{N}$, the lead slope of g_{i+1} in A is less than the lead slope of g_i in A .*
2. *Given any real number $q > 1$, there is an $i \in \mathbf{N}$ so that the lead slope of g_i in A is p where $1 < p < q$.*

Then there is $c \in (a, b)$ so that given any real number $s > 1$, H has an element α which has an affine component Γ containing (a, c) , and with slope r on Γ where $1 < r < s$.

pf:

To simplify the language of this argument, we will restrict our attention to the orbital A , treating it as the domain of the elements of H , so that the phrase “The first affine component of $[h \in H]$ ” will really mean the open interval (a, u) where h has an affine component of the form (v, u) where $v \leq a < u$. We will also refer to this as the “leading (or lead) affine component of h ”. Similarly, we will refer to the slope of h on its leading affine component as the “lead slope of h ”.

Note that the second condition on $(g_n)_{n=0}^\infty$ implies that every element of $(g_n)_{n=0}^\infty$ has lead slope greater than one.

For each $i \in \mathbf{N}$, let (a, b_i) be the first affine component of g_i , and let s_i represent the lead slope of g_i .

We are now in a position to define a new sequence of functions $(h_i)_{i=0}^\infty$ which satisfy the following conditions:

1. For each $i \in \mathbf{N}$, the lead slope of h_i is s_i .
2. For each $i \in \mathbf{N}$, the leading affine component of h_i contains (a, b_0) .

Given such a sequence $(h_i)_{i=0}^\infty$ and $s > 1$, the hypotheses on the g_i show that there will be an $N \in \mathbf{N}$ so that for all $n > N$ we will have that $s > s_n > 1$. If we further set $c = b_0$, then for all $n \in \mathbf{N}$, h_n will have leading affine component containing (a, c) , so we will have finished the proof of the lemma.

For each $i \in \mathbf{N}$, define n_i to be the smallest non-negative integer so that $b_0 g_0^{-n_i} < b_i$, and define $h_i = g_0^{-n_i} g_i g_0^{n_i}$. Since g_0 moves points to the right on its first affine component, n_i is well defined, and therefore h_i is well defined. We now check that h_i satisfies the two

conditions, for each i . Firstly, we observe that the lead slope of each h_i is the product of the lead slopes of the elements of the product $g_0^{-n_i} g_i g_0^{n_i}$, which is $(\frac{1}{s_0})^{n_i} s_i s_0^{n_i} = s_i$. Secondly, by Remark 3.3, we know that if $x \in (a, b)$ is a breakpoint of the product $g_0^{-n_i} g_i g_0^{n_i}$, then the image of x under application of some initial partial product (possibly empty) must be a breakpoint of the next term of the overall product. However, the first breakpoint of g_0^{-1} is $d_0 = b_0 g_0 > b_0$, and g_0^{-1} moves points left in its first affine component so if $x \in (a, b_0)$, then $x g_0^{-k} < d_0$ for all non-negative integers k . In particular, if $x \in (a, b_0)$, then x is in the first affine component of $g_0^{-n_i}$ for any $i \in \mathbf{N}$. Given an $i \in \mathbf{N}$, the first breakpoint of g_i in A is b_i , but n_i was chosen so that $b_0 g_0^{-n_i} < b_i$, so if $x \in (a, b_0)$, then the image of x under $g_0^{-n_i}$ is in the first affine component of g_i , in particular, (a, b_0) is contained in the first affine component of $g^{-n_i} g_i$. Finally, the first breakpoint of $g_0^{n_i}$ is the image of b_0 under $g_0^{-n_i+1}$, but $b_0 g_0^{-n_i} g_0 \geq b_0 g_0^{-n_i} g_i$ because the leading slope of g_i is less or equal to the leading slope of g_0 . In particular, the whole interval (a, b_0) is in the first affine component of h_i . \diamond

The following lemma expresses the main point of the section, as it will enable us to find the remarkable “controllers”; elements that control the global behavior of a balanced group on its orbitals.

Lemma 3.11 *Suppose H is a balanced group with single orbital $A = (a, b)$, and ϕ is the log-slope homomorphism defined before, then $\phi(H) \cong \mathbb{Z}$ or $\phi(H)$ is trivial in $\mathbf{R} \times \mathbf{R}$.*

pf: Let H be a balanced subgroup of $PL_o(I)$. Each element of H either realizes both ends of A , or neither. In particular, the group homomorphism $\rho_1 : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ which is projection on the first factor has the property that $\ker(\rho_1) \cap \text{Im}(\phi) = \{(0, 0)\}$, the trivial subgroup of $\mathbf{R} \times \mathbf{R}$. This implies that the image of ϕ in $\mathbf{R} \times \mathbf{R}$ is isomorphic to the group $H_1 \leq \mathbf{R}$ obtained by considering only the first factors of elements of $\text{Im}(\phi)$. If no element realizes the ends of A , then $H_1 = \{0\}$, the trivial (additive) group, and we are done. Therefore, let us suppose instead that some elements in H realize the left end of A (and therefore also the right) so that H_1 cannot be the trivial subgroup of \mathbf{R} .

If H_1 is discrete in \mathbf{R} then H_1 is either trivial, or isomorphic to \mathbf{Z} , but by assumption, H_1 is not the trivial group, hence in this case $H_1 \cong \mathbf{Z}$. Hence, we shall suppose that H_1 is not discrete in \mathbf{R} . In this case, by taking the difference of two elements in the image which are very near each other, we see that we can find an element of H_1 which is as close to zero as we like. This implies there are elements of H whose leading slopes are as close to one as we like, without actually being one. If h is an element with leading slope $s \neq 1$, then one of h or h^{-1} has slope greater than one, since the leading slope of h is s , but the leading slope of h^{-1} is $1/s$ (note that s cannot be zero, since no element of $PL_o(I)$ has an affine component with slope zero).

Now suppose that H is abelian. By a result of Brin and Squier in [6], if two elements in $PL_o(I)$ commute and share a common orbital W , then there is an element $w \in PL_o(I)$ which behaves as a the identity off of W and is a common root of the initial two elements over W . If two elements have non-disjoint support and commute, it is easy to see that the intersections of their supports actually is a set of commonly shared orbitals. Now, let h be some element of H with leading slope $s > 1$. For each positive integer n , let g_n be an element of H with leading slope s'_n where $1 < s'_n < \frac{n+1}{n}$. Now for each g_n , the pair h and g_n has a common root h_n (on their leading orbital), but infinitely many of the roots h_n have pairwise

distinct leading slopes, since these slopes are always less than or equal to the slopes of the g_n , and in particular, h must then have infinitely many distinct roots in H on its leading orbital. By another result in [6], no element of $PL_o(I)$ has infinitely many distinct roots, so we must conclude that H is not abelian.

Note that by the details of the previous paragraph, it is easy to construct a countably infinite sequence of elements $(g_n)_{n=1}^\infty$ in H which satisfies the properties below:

1. For each $i \in \mathbf{N}$, the lead slope of g_{i+1} is less than the lead slope of g_i .
2. Given any real number $r > 1$, there is an $i \in \mathbf{N}$ so that the lead slope of g_i is s where $1 < s < r$.
3. given $i, j \in \mathbf{N}$, we will have $[g_i, g_j] = 1$ implies $i = j$.

Therefore, by Lemma 3.10 there is $c \in (a, b)$ and elements of H with lead slopes that are greater than but arbitrarily close to one, and whose leading affine components contain (a, c) .

Let f and g be two elements of H with $h = [f, g] \neq 1$. The fixed set of h in I is disconnected, and contains two components of the form $[0, u']$ and $[v', 1]$ for some numbers $u', v' \in (a, b)$. In particular, $\inf(\text{Supp}(h)) = u'$ and $\sup(\text{Supp}(h)) = v'$.

By Lemma 3.4 there is $\alpha \in H$ so that $v'\alpha < c$, so that $j = h^\alpha$ has all of its orbitals inside (a, c) . Let $u = \inf(\text{Supp}(j))$ and $v = \sup(\text{Supp}(j))$. In particular, $v < c$.

We will now perturb j slightly via a conjugation which will move the orbitals of j to the right by a distance less than L , so that j and the new element together will generate a group with an imbalanced orbital.

Suppose $L > 0$ is smaller than two particular lengths. The first length is the length of the second component of the fixed set of j in A which has non-zero length (note, this component might just be $[v, b]$, if j has only two such), and the second length is the length of the first orbital of j .

Choose an element $\beta \in H$ whose leading affine component in A contains (a, c) and whose lead slope is greater than one, but so near one that no point in (a, c) will move to the right a distance greater than L . Now the elements j and j^β will generate a group G with leading orbital (u, w) where j realizes u and possibly w (if the right ends of the appropriate orbitals of j and j^β are aligned), while j^β will achieve w but not u . To see this, note that the left end of the first orbital of j^β is in the first orbital of j , so that u is the left end of the leading orbital of G , and only j realizes it. Meanwhile, the right end of the first orbital of j^β is to the right of the right end of the first orbital of j , so that the leading orbital of j^β contains the right end of the leading orbital of j . As we progress to the right, if the solitary fixed point sets of j and j^β align before we reach the second component of the fixed set of j in A with non-zero length, then the right end of the first orbital of G will be achieved by both j and j^β . Otherwise, the first orbital of G will extend rightward into the interior of the second fixed set of j with a non-empty interior, so that the right end of this orbital of G will be realized only by j^β . In all cases, G will be imbalanced, and hence H will also be imbalanced. This contradicts the hypotheses of the lemma, and so we see that H_1 must be discrete in \mathbf{R} , and therefore the image of ϕ is isomorphic to \mathbf{Z} or the trivial group in $\mathbf{R} \times \mathbf{R}$. \diamond

The kernal of the homomorphism ϕ is naturally very important as well, it is the subgroup of H which consists of elements which are the identity near the ends of A . Typically, we will refer to this normal subgroup as \mathring{H} .

3.3.2 Controllers

A consequence of section 3.3.1 is that the structure of a balanced group with one orbital is very special. In this section we will explore this idea further.

Lemma 3.12 (Balanced Generator Existence) *Suppose that H is a balanced subgroup of $PL_o(I)$ with single orbital A , that there is some element in H which realizes an end of A , and that \mathring{H} is the subgroup of H which consists of all elements in H which are the identity near the ends of A . Then there is an element g of H so that $H = \langle \langle g \rangle, \mathring{H} \rangle$, where g realizes both ends of A .*

pf:

Let Γ_A be the set of elements of H which realize both ends of A . By our assumptions, Γ_A is not empty. Now observe that $H = \langle \langle \Gamma_A \rangle, \mathring{H} \rangle$.

By lemma 3.11 the image $\phi(H)$ is infinite cyclic in $\mathbf{R} \times \mathbf{R}$. Let γ be a generator of the image of ϕ . Let g be an element of H so that $g\phi = \gamma$. We observe that since γ is non-trivial in both components, g realizes both ends of A . Since γ generates the image of ϕ , if $\hat{g} \in \Gamma_A$, then $\hat{g}\phi = g^k\phi$ for some $k \in \mathbf{Z}$. Hence, $g^{-k} \cdot \hat{g} \in \mathring{H}$. This now implies that $\Gamma_A \subset \langle g, \mathring{H} \rangle$, so that $\langle g, \mathring{H} \rangle = \langle \Gamma_A, \mathring{H} \rangle = H$. \diamond

We will call any element c of a balanced group H with one orbital A , which satisfies the rule $H = \langle \langle c \rangle, \mathring{H} \rangle$, a *controller* of H . A controller of H is clearly a special element.

Given a controller c of a balanced group H with one orbital A , we can write any element h of H uniquely in the form $c^k \cdot \mathring{g}$, where k is some integer, and $\mathring{g} \in \mathring{H}$. We will call this the *c-form* of h .

We will say that a controller c of the group H is *consistent* if and only if its image $(\alpha, \beta) = c\phi$ satisfies the property that $\text{sign}(\alpha) = -\text{sign}(\beta)$. Otherwise we will say the controller is *inconsistent*. The idea behind this definition is that a one orbital controller should be consistent, since it is either moving points to the right everywhere on its support, or it is moving points to the left everywhere on its support. An inconsistent controller must have a fixed point set, and has at least one orbital where the controller moves points to the right, and one orbital where the controller moves points to the left. It turns out that a consistent controller actually is a one-orbital element of H .

Lemma 3.13 *Suppose H is a balanced subgroup of $PL_o(I)$ and H has unique orbital A . Further suppose that H has a consistent controller c , then c realizes A .*

pf:

Since c and c^{-1} are both controllers, and either both satisfy or both fail the conclusion of the statement of the lemma, we will assume c moves points to the right on its orbitals near the ends of A . Suppose c has a non-trivial fixed set K in A . K is closed and bounded and hence compact. Let $u = \inf K$ and $v = \sup K$, so that (a, u) is the first orbital of c and (v, b) is the last orbital of c . Now there are points $x \in (a, u)$, and $y' \in (v, b)$ so that we have $a < x < u \leq v < y' < b$. By Lemma 3.4 there is an element $g \in H$ so that $xg > y'$. Writing g in its c -form, we have that $g = c^k \mathring{g}$ for some integer k and element $\mathring{g} \in \mathring{H}$. In

particular, the element $h = gc^{-k}$ is trivial near the ends of A , but satisfies $xh = y > v$. The element h therefore has an orbital $D = (r, s)$ which spans the fixed set K of c . Suppose $e = \inf(\text{Supp}(h))$, so that $a < e \leq r$. By Lemma 3.1 there is a positive integer N_1 so that for any integer $n_1 > N_1$ we will have $r < ec^{n_1} < u$. Suppose $f = \sup(\text{Supp}(h))$, so that $s \leq f < b$. By Lemma 3.1 there is a positive integer N_2 so that for any integer $n_2 > N_2$ we will have $f < sc^{n_2} < b$. Let $n = \max(N_1, N_2)$, then, the element $j = h^{(c^n)}$ has its first orbital starting at some interior point of (r, s) , and its orbital induced from (r, s) has right end t which is strictly to the right of f . In particular the group $H_1 = \langle j, h \rangle$ has an orbital $B = (r, t)$, where we note that $t > s$ by construction. Now, h realizes the left end of B in B , but not the right, hence H_1 , and therefore H , is imbalanced. But this contradicts our assumptions, therefore c must have orbital A . \diamond

The following corollary is left to the reader.

Corollary 3.14 *Suppose H is a balanced subgroup of $PL_o(I)$ and H has a unique orbital $A = (a, b)$.*

1. *If H has a consistent controller, c , then any element g of H which realizes both ends of A actually realizes A .*
2. *If H has an inconsistent controller c , then no element of H realizes A .*

Suppose that we know that A is an orbital of a subgroup $H \leq PL_o(I)$. Let H_A be the set of elements of $PL_o(I)$ each of which is equal to the restriction of some element of H on the orbital A , and behaves as the identity off of A . H_A is trivially a group with unique orbital A , and is a quotient of H . We will call H_A the *projection of H on A* .

We will now generalize our language somewhat. Let H be a subgroup of $PL_o(I)$ with an orbital A , and let H_A be the projection of H on A . H_A has a controller \tilde{c} for A . Let $\rho : H \rightarrow H_A$ be the projection homomorphism on the orbital A . Let

$$T = \{c \in H \mid c\rho \text{ is a controller for } H_A \text{ on } A\}.$$

We will call any element of T a *controller of H on A* . Again, given an element $c \in H$ which is a controller of H on A , we can write elements of H in a unique c -form. I.e., if $g \in H$, and c is a controller for H on A , then there is an integer k so that $g = c^k \mathring{g}$, where \mathring{g} will not realize either end of A .

4 Connecting the Two Algebraic Descriptions

In this section, we will complete the proofs of our two main theorems by introducing a new geometric technique. Our technique allows us to embed a solvable group of derived length n in $PL_o(I)$ into a more tractable group in $PL_o(I)$ with derived length n .

4.1 Technical Preliminaries

The next lemma is a technical lemma that we will use in completing our proof of Theorem 1.1.

Lemma 4.1 *If G is a solvable subgroup of $PL_o(I)$ with derived length n , generated by a collection Γ of elements of $PL_o(I)$ which each admit exactly one orbital, and where no generator can be conjugated by an element of G to share an orbital with a different generator, then G is isomorphic to a group in the class $\{1\}\mathcal{P}$ with derived length n .*

pf: Before getting into the main body of the proof, note that the hypotheses imply the main points of Lemma 3.7, namely, that G is balanced and admits no transition chains of length two. Further, the hypotheses further imply that each generator is the only generator with that orbital, and that no element orbital is a union of element orbitals that do not realize the original orbital.

We now enter the main body of the proof. We will proceed by induction on n .

If $n = 0$ then G is the trivial group, and $G \in \{1\}\mathcal{P}$. If $n = 1$ then G is abelian, and in particular, there can be at most countably many generators in Γ , all of which have disjoint support, so that G is isomorphic with a countable (or finite) direct sum of \mathbf{Z} factors, so $G \in \{1\}\mathcal{P}$.

Now let us suppose that $n > 1$ and that the statement of the lemma is correct for any such solvable group with derived length $n - 1$. Let X represent the generators in Γ whose orbitals are all depth 2 in G or deeper. Let Y be the set of elements in Γ whose orbitals have depth 1. We note in passing that the cardinality of Y is at most countably infinite, and that the collection of orbitals of the elements of Y actually form the orbitals of the group G . We will assume that all of the elements in Y move points to the right on their orbitals. We can partition the elements in X into sets P_y , where the y index runs over the elements in Y , and where an element of X is in P_y if and only if that element's orbital is contained inside the orbital of y . Given $y \in Y$, if P_y is empty, then define $H_y = \langle y \rangle$. Otherwise, let $\gamma \in P_y$, and suppose that γ has smallest depth possible for the elements in P_y , and that γ has orbital $A = (a, b)$. y has a fundamental domain $D_y = [a, ay)$, and each element of P_y may be conjugated by some power of y so that the resultant element's orbital lies in the fundamental domain D_y (if some element, β , conjugates to contain a in its orbital, then either that conjugate has that its orbital fully contains the orbital A , which is impossible by our choice of γ as having a minimal depth orbital of the orbitals of all the elements in P_y , or the signed orbitals of γ and of the conjugate of β form a transition chain of length two, which is impossible since G is solvable). We can now replace P_y by the conjugates of the original P_y found above so each element of P_y has its orbital in $[a, ay)$, and the group generated by the new P_y with y will be identical to the group generated by the old P_y with y . However, now that all of the elements of P_y have supports in the same fundamental domain of y , we have that $H_y = \langle P_y, y \rangle$ is isomorphic to $K_y \wr \mathbf{Z}$, where $K_y = \langle P_y \rangle$. But K_y is a solvable group of precisely the type mentioned in the hypotheses of the lemma, with derived length k less than n , so that K_y is isomorphic to a group in $\{1\}\mathcal{P}$ with derived length $k < n$. Therefore, H_y is isomorphic to a group in $\{1\}\mathcal{P}$ (being the result of a group in $\{1\}\mathcal{P}$ being wreathed with a \mathbf{Z} factor on the right) with derived length $k + 1 \leq n$. This argument holds for every y in Y , so $G \cong \bigoplus_{y \in Y} H_y$, where all of the groups in this countable direct sum have derived length less than or equal to n (and at least one of them has derived length n). In particular, we see that G is isomorphic to a group in $\{1\}\mathcal{P}$ with derived length n . \diamond

4.2 The Split Group

One new tool for technical analysis of a subgroup G of $PL_o(I)$ is the split group of G . It is motivated by the hypotheses of Lemma 4.1. We define the split group of a subgroup of $PL_o(I)$ below.

Let P_s represent the set of subgroups of $PL_o(I)$. We define a function $S : P_s \rightarrow P_s$. Given a group G which is a subgroup of $PL_o(I)$, fix the notation Γ_G to represent the maximal set of one-orbital elements of $PL_o(I)$ such that if $\gamma \in \Gamma_G$, then γ is identical to an element $g \in G$ over γ 's orbital. Define S by the rule that $S(G) = \langle \Gamma_G \rangle$ (if G is the trivial group, define $S(G) = G$). Given a group $G \in P_s$, we will call the group $S(G)$ the *split group of G* . Note that $G \leq S(G)$. In this section, we will analyze further properties of the split group $S(G)$, in the case that G is solvable.

Throughout the remainder of the section, given a group G , and any $\alpha \in \Gamma_G$, let A_α always denote the orbital of α . Below, we mention some further basic facts relevant to the analysis of the split group of a group, some using the notation just established. We leave the proofs to the reader.

Remark 4.2 *Suppose G is a solvable subgroup of $PL_o(I)$, then*

1. *The depth of any signed orbital of G is a positive integer.*
2. *If $A = (a, b) \subset I$, and both $g, h \in G$ have orbital A , then the depths of the signed orbitals (A, g) and (A, h) of G are the same.*
3. *If (A, g) is a signed orbital of G with depth k , and $h \in G$, then (Ah, g^h) is a signed orbital of G with depth k .*
4. *If $\alpha, \beta \in \Gamma_G$ and the orbital of α is contained in the orbital of β , then α^β is in Γ_G .*
5. *If $\alpha, \beta \in \Gamma_G$, then $\{(A_\alpha, \alpha), (A_\beta, \beta)\}$ is not a transition chain of length two for $S(G)$.*

Note that in the case where G is solvable, the first and second points above allow us to define a G -depth for elements in Γ_G . Namely, if $\gamma \in \Gamma_G$ and the orbital of γ is A , then there is a positive integer k so that all signed orbitals of the form (A, h) for G (in particular, this includes all of the signed orbitals where the element h agrees with γ over A) have depth k , so we can say γ has G -depth k .

The following lemma follows easily from the definition above, and points 4 and 5 of the previous remark. It provides us with something like a normal form for writing elements of $S(G)$ in terms of the generators in Γ_G .

Technical Lemma 4.3 *Suppose G is a solvable subgroup of $PL_o(I)$ and $h \in S(G)$, then given a product decomposition $h = \tau_1 \tau_2 \cdots \tau_k$ so that each τ_i is an element of Γ_G for each positive integer $i \leq k$, then there is a product decomposition $h = \theta_1 \theta_2 \cdots \theta_k$ with each $\theta_i \in \Gamma_G$, so that whenever r and s are positive integers so that $r < s \leq k$, we have the G -depth of θ_s is greater than or equal to the G -depth of θ_r .*

pf:

We will show that if $h \in S(G)$, and $h = \alpha_1 \alpha_2$ with $\alpha_1, \alpha_2 \in \Gamma_G$, then we can write $h = \theta_1 \theta_2$ with θ_1 and $\theta_2 \in \Gamma_G$ and with the G -depth of θ_2 greater than or equal to the G -depth of θ_1 , and with the set of G -depths of the θ_i 's the same as the set of G -depths of the α_i 's. We then will have our lemma by re-writing longer products, improving the product locally on adjacent pairs of length two until there are no more improvements to be made.

Let us assume the G -depth of α_1 is greater than the G -depth of α_2 , as the other cases are trivial. In this case let $\theta_1 = \alpha_2$ and let $\theta_2 = \alpha_1^{\alpha_2}$. If α_2 and α_1 have orbitals that intersect, then point 5 from Remark 4.2 and the definition of G -depth guarantee that the orbital of α_1 is contained in the orbital of α_2 , and then point 4 of Remark 4.2 gives us that $\theta_2 \in \Gamma_G$. In the case that the orbitals of α_1 and α_2 are disjoint, then $\theta_2 = \alpha_1 \in \Gamma_G$, so in all cases θ_1 and $\theta_2 \in \Gamma_G$. Now we can afford the cost of pushing α_2 past α_1 :

$$h = \alpha_1 \alpha_2 = \alpha_2 \cdot \alpha_2^{-1} \alpha_1 \alpha_2 = \theta_1 \cdot \theta_2$$

Note that the G -depth of θ_1 is the G -depth of α_2 and the G -depth of θ_2 is the G -depth of α_1 (by a variant of point three from Remark 4.2).

◇

We will need one more technical lemma before we can come to some meaningful conclusions about the split group of a solvable group G .

Technical Lemma 4.4 *Suppose G is a solvable subgroup of $PL_o(I)$. If $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ is a collection of elements of Γ_G for some integer k , and for each integer j with $2 \leq j \leq k$ we have that $\overline{A}_{\alpha_j} \subset A_{\alpha_1}$, then the product $\tau_{1,k} = \alpha_1 \alpha_2 \cdots \alpha_k$ will have support A_{α_1} .*

pf:

Call a finite product of elements of Γ_G a *first orbital dominant product* if the closure of each orbital of the later terms in the product is contained in the orbital of the first term of the product.

It is immediate that the support of a first orbital dominant product is contained in the orbital of the first element of the product. We will now show that such a product admits no fixed points in the orbital of the first element of the product.

Let us suppose first that k is the smallest integer so that a first orbital dominant product of k elements of Γ_G has a fixed point in the orbital of the first element of the product, and let $\alpha_1 \alpha_2 \cdots \alpha_k$ be such a product. We will derive a contradiction, and therefore we will be able to conclude that any first orbital dominant product will have the whole of the orbital of the first element in the product as its support.

We note in passing that $k > 1$.

By reference to Technical Lemma 4.3, we assume (without risking the first orbital dominant nature of the product) that if $i < j$ are integers with $1 \leq i < j \leq k$ then the G -depth of α_i is less than or equal to the G -depth of α_j . Let us build some notation for subproducts of our newly re-arranged α_i 's. Given any integers i and j so that $1 \leq i \leq j \leq k$ let $\tau_{i,j}$ represent the product $\alpha_i \alpha_{i+1} \cdots \alpha_j$.

If $x \in A_{\alpha_1}$ so that $x \tau_{1,k} = x$, then since x is not a fixed point of $\tau_{1,k-1}$ we see that $x \in A_{\alpha_k}$. Since the fixed set of a product of elements of $PL_o(I)$ is closed, let us further assume that x is the infimum of the points in A_{α_1} that are fixed by $\tau_{1,k}$. Denote x by y_0 ,

and further, let $y_j = y_0\tau_{1,j}$ for each integer j with $1 \leq j \leq k$, so that, for example, $y_k = x$. If there is a positive integer $j \leq k$ with $y_j = y_{j-1}$ then the product $\alpha_1\alpha_2 \cdots \alpha_{j-1}\alpha_{j+1}\alpha_{j+2} \cdots \alpha_k$ will also have fixed point x , contradicting the minimality of k , so we see that for each positive integer j with $j \leq k$, $y_{j-1} \neq y_j$. It is immediate from the last condition that for any j with $0 \leq j < k$, $y_j \in A_{\alpha_{j+1}}$. The previous sentence implies that for each i with $1 \leq i < k$ we have $A_{\alpha_{i+1}} \cap A_{\alpha_i} \neq \emptyset$. Therefore, $A_{\alpha_{i+1}} \subset A_{\alpha_i}$ since the G -depths of the α_i are non-decreasing and G admits no transition chains of length two.

By the definition of Γ_G , for each integer i with $1 \leq i \leq k$ there are elements $g_i \in G$ with α_i behaving like g_i over A_{α_i} . We observe that the product $g = g_1g_2 \cdots g_k$ will also fix x , since $y_{i-1} \in A_{\alpha_i}$ for all integer i with $1 \leq i \leq k$. Denote by a_k the left hand endpoint of A_k . We will show that $a_k g \neq a_k$, implying that g has an orbital containing the left end of the orbital A_{α_k} of g_k .

We observe immediately that $(a_k, x) \subset A_{\alpha_{k-1}}$. Define $b_k = a_k$ and inductively define, for each integer j with $0 \leq j < k$ the sets $(b_j, y_j) = (b_{j+1}, y_{j+1})\alpha_{j+1}^{-1}$. Since $(b_j, y_j) \subset A_{\alpha_{j+1}}$ for all $j \in \mathbf{N}$ with $j < k$, we see that g agrees with $\tau_{1,k}$ over (b_0, y_0) , and therefore by the continuity of g and $\tau_{1,k}$, these two maps agree over $[b_0, y_0]$. But $y_0 = x$, where g is fixed, while $b_0\tau_{1,k} = a_k$. Since $\tau_{1,k}$ is not fixed at b_0 (this follows from the facts that $k > 1$ and $\bar{A}_j \subset A_1$, for $j \in \mathbf{N}$ with $2 \leq j \leq k$), g moves a_k . In particular, g has an orbital B containing a_k but not x , so that $\{(B, g), (A_k, g_k)\}$ is a transition chain of length two for G , which contradicts the solvability of G .

◇

We are now in a position to analyze the depth of the split group of a solvable group G .

Lemma 4.5 *Suppose G is a solvable subgroup of $PL_o(I)$ that has derived length u for some $u \in \mathbf{N}$. If (A, h) is a signed orbital of $S(G)$, then there is $g \in G$ so that (A, g) is a signed orbital of G .*

pf:

If $u = 0$ there is nothing to prove, so let us assume that $u > 0$, so that $S(G)$ is not the trivial group. In particular, we assume that $S(G)$ has some associated signed orbitals.

Suppose (A, h) is a signed orbital of $S(G)$, and let $h = \alpha_1\alpha_2 \cdots \alpha_k$ for some positive integer k , where each α_i is in Γ_G . In this argument, we will give an algorithm that steadily improves h and the product expression for h (always preserving the fact that h has orbital A). After each improvement, we will assume the product is re-indexed as in the product above, so that, for instance, k will always refer to the length of the current product. At some point, we will see that G has an element with orbital A .

Our algorithm begins in the next paragraph. At some points we improve the product definition of h , and we then begin the algorithm again. This could easily lead to infinite loops in our algorithm. After the algorithm is fully specified, we will argue that the direction to restart the algorithm, given in the algorithm's final paragraph, avoids the creation of such an infinite loop. We will leave the argument for termination of the algorithm for each of the previous directions to re-start to the reader.

Algorithm Initiation

1. Assurance that k is large.

If $k = 1$ then $A_{\alpha_1} = A$ and G has an element g which behaves as α_1 on A . Therefore let us assume that $k > 1$.

2. G -depth ordering, and some notation.

By Technical Lemma 4.3 we can assume that the product decomposition for h has the property that if r and s are positive integers so that $r < s \leq k$, then the G -depth of the orbital of α_s is greater than or equal to the G -depth of α_r .

Given integers i and j with $1 \leq i \leq j \leq k$ set $\tau_{i,j} = \alpha_i \alpha_{i+1} \cdots \alpha_j$, so that $h = \tau_{1,k}$.

3. $A \subset A_{\alpha_1}$.

Suppose A_{α_1} is disjoint from A . In this case $\tau_{2,k}$ must have orbital A . Replace h by $\tau_{2,k}$, and start the algorithm again using the one-shorter product description of the new h , re-indexed. We can assume from here out that $A \cap A_{\alpha_1} \neq \emptyset$. We will refer to A_{α_1} , as determined at this stage, to be the *leading orbital of the product*.

It is immediate from the definition of G -depth that if r, s are integers so that $1 \leq r < s \leq k$ then either $A_{\alpha_s} \subset A_{\alpha_r}$ or these two element orbitals are disjoint (recall point 5 of Remark 4.2).

We can strengthen the result of the paragraph two paragraphs back by noting that if A is not contained in A_{α_1} , then some end e of A_{α_1} must be in A . In this case e is moved by one of the α_i for an integer i with $2 \leq i \leq k$. This last is impossible by the previous paragraph, therefore $A \subset A_{\alpha_1}$.

4. Forcing $A_{\alpha_i} \subset A_{\alpha_1}$ for $1 \leq i \leq k$.

Now, for any integer i with $2 \leq i \leq k$, the orbital A_{α_i} must either be contained in A_{α_1} or must be disjoint from A_{α_1} . In particular, if we set $C_j = A\tau_{1,j}$ for each integer j with $1 \leq j \leq k$, we see that for each such j , $C_j \subset A_{\alpha_1}$. But this implies that we can drop any α_j from the product producing h , if that α_j has support disjoint from A_{α_1} , and the resulting product will still act as h over A . If there are any such j , let us drop the corresponding α_j from the product, re-index, and begin the algorithm again (to catch the case that the resulting product has length one, for instance). If there are no such j , then we see that for each integer j with $2 \leq j \leq k$ we must have $A_{\alpha_j} \subset A_{\alpha_1}$.

5. Finding how many of the A_{α_i} are equal to A_{α_1} .

For each integer i with $1 \leq i \leq k$ note that there is an element $g_i \in G$ so that g_i behaves as α_i over the interval A_{α_i} .

Suppose that for some integer i in $2 \leq i \leq k$ we have that A_{α_i} shares an end with A_{α_1} . Since G is balanced, either $A_{\alpha_i} = A_{\alpha_1}$ or g_i has at least two orbitals contained in A_{α_1} , one sharing one end of A_{α_1} , and the other sharing the other end of A_{α_1} . It is easy in the second case to show that G admits transition chains of length two. In particular, we can now assume that for any integer i where $2 \leq i \leq k$, the orbital A_{α_i} either equals A_{α_1} or has closure contained in A_{α_1} .

By the fact that the G -depth of the elements α_i is non-decreasing in i , we see that there is an integer s with $1 \leq s \leq k$ so that for all integers $1 \leq r \leq s$ we have $A_{\alpha_r} = A_{\alpha_1}$ while for all integers t with $s < t \leq k$ we see that $\overline{A_{\alpha_t}} \subset A_{\alpha_1}$.

If $s = 1$, Technical Lemma 4.4 assures us that $A = A_1$. In this case, g_1 has orbital A and we are finished. Therefore, let us assume that $s > 1$, so that $A_{\alpha_1} = A_{\alpha_2}$.

6. Collapsing the product $\alpha_1\alpha_2$ over A_{α_1} .

If $\tau_{1,2}$ is the identity, then $\tau_{3,k}$ behaves as h over A . In this case, remove α_1 and α_2 from the product, re-index, and begin the algorithm again. If $\tau_{1,2}$ is not the identity, continue below.

Let $\{B_1, B_2, \dots, B_m\}$ be the orbitals of $\tau_{1,2}$. These orbitals are all contained in A_{α_1} . Note that $\tau_{1,2}$ behaves as an element of G over A_1 . In particular, there are elements $\beta_i \in \Gamma_G$ for each integer $1 \leq i \leq m$ so that β_i behaves as $\tau_{1,2}$ over B_i .

Let $\Upsilon = \{i \in \mathbf{N} \mid 1 \leq i \leq m, B_i \cap A \neq \emptyset\}$. Form the product $\omega = (\prod_{i \in \Upsilon} \beta_i)\alpha_3\alpha_4 \cdots \alpha_k$, which is a product of elements of Γ_G which behaves as h over A , but has fewer elements in the product with orbital A_{α_1} . Begin the algorithm again using the product definition of ω to define the new α_i , indexed appropriately.

Algorithm Termination

We will now argue that the algorithm given above terminates, in terms of the final paragraph's re-direction to repeat the algorithm. The previous re-directions cannot create an infinite loop for straightforward reasons which we leave to the reader.

The final paragraph of the algorithm either decreases the number of elements in the product which have orbital equal to the leading orbital of the product (which can only happen a finite number of times for any particular leading orbital, bounded by $s - 1$ for the initial s of that leading orbital), or it will decrease the size of the leading orbital. If the algorithm decreases the size of the leading orbital u times then G will admit a tower of height $u + 1$, which is impossible as G has derived length u .

The previous paragraph implies that the algorithm must terminate before the steps in the final paragraph of the algorithm description decrease the size of the leading orbital for the u 'th time, which means that it will have found an element of G with orbital A .

◇

Corollary 4.6 *Suppose G is a subgroup of $PL_o(I)$. The derived length of G equals the derived length of $S(G)$.*

pf:

If G is solvable then Lemma 4.5 applies; given any tower T of $S(G)$ we can find a tower with the same orbitals for G . Hence, the derived length of $S(G)$ is less than or equal to the derived length of G . But since $G \leq S(G)$ we see that the derived length of G is automatically less than or equal to the derived length of $S(G)$. In particular, if G is solvable, these derived lengths are equal.

If G is non-solvable, then both G and $S(G)$ have towers of arbitrary height, so that both groups are unsolvable.

◇

4.3 Split Groups and \mathcal{M}

The following lemma, and its corollary, complete our proof of both Lemma 2.3 and Theorem 1.1. Note that the corollary is simply a restatement of Lemma 1.3.

Lemma 4.7 *If G is a solvable subgroup of $PL_o(I)$, then $S(G)$ is isomorphic to a group in $\{1\}\mathcal{P}$.*

pf:

Suppose that $n \in \mathbf{N}$ and that G is a solvable subgroup of $PL_o(I)$ with derived length n . By Corollary 4.6 we know that the derived length of $S(G)$ is also n . By Lemma 3.7 we see that G admits no transition chains of length 2.

Let X_1 represent the set of signed orbitals of H with depth 1. For each orbital A in O_{X_1} , there is a non-empty set of controllers of H for A in the set Γ . Let $\phi_1 : O_{X_1} \rightarrow \Gamma$ represent a function that associates to each orbital in O_{X_1} a controller of H for that orbital which moves points to the right on the orbital. Let $Y_1 = O_{X_1}\phi_1$ be the image of ϕ_1 . We note that each pair of elements in Y_1 have disjoint support, and trivially, that no element of Y_1 can be conjugated by an element of H to share an orbital with a different element of Y_1 . Now, by the definition of controller, Y_1 consists of a set of generators in Γ sufficient so that the set $\Gamma_1 = Y_1 \cup (\Gamma \setminus S_{X_1})$ generates H .

For each $y \in Y_1$, let A_1^y represent the orbital of y . We may partition the elements of Γ_1 into sets P_1^y indexed by the set Y_1 so that $\gamma \in P_1^y$ if the orbital of γ is contained in A_1^y . Now let X_2^y represent the set of signed orbitals of elements in P_1^y with depth two in H . If X_2^y is not empty, let (A_2^y, γ_2^y) be an element in X_2^y , and let a_y be the left end of the orbital A_2^y . Since the orbitals of elements of H are always orbitals of G by Lemma 4.5, we see that H admits no transition chains of length two. In particular, we can use y to conjugate every signed orbital of X_2^y into the fundamental domain $[a_y, a_y y)$ to produce the set D_2^y of signed orbitals. Likewise, we can conjugate every signed orbital in P_1^y of depth greater than two into the fundamental domain as well, producing the set T_2^y . The collections of conjugates in the fundamental domain have nice properties:

1. If two elements of D_2^y have orbitals that non-trivially intersect each other, then they actually have identical orbitals.
2. $H_{A_1^y} = \langle y, S_{D_2^y}, S_{T_2^y} \rangle$.

Let $\phi_2^y : O_{D_2^y} \rightarrow S_{D_2^y}$ be a function that picks for each orbital of depth two in $O_{D_2^y}$ a controller that moves points to the right for that orbital as before. Let Y_2 be the union of all the images of the functions ϕ_2^y across the Y index set, so that we have now picked a controller that moves points to the right on its orbital for every conjugacy class of depth two orbitals of H (note that a conjugate of a controller is also a controller), so that if $\Gamma_2 = (\Gamma_1 \setminus (\cup_{y \in Y} S_{X_2^y})) \cup Y_2$ then $H = \langle \Gamma_2 \rangle$.

In a like fashion we can inductively proceed to pick sets of controllers, one for each conjugacy class of element orbital of depth i , where i is an index less than or equal to n , in exactly the same fashion as discussed for forming the set Y_2 above. This process will steadily improve the sets of generators Γ_i , so that finally Γ_n will be a set of generators for H where

each generator in Γ_n has exactly one orbital, and where no generator can be conjugated in G to share an orbital with another generator in Γ_n . In particular, by Lemma 4.1, H is isomorphic to a group in $\{1\}\mathcal{P}$ with derived length n , and therefore G is isomorphic to a subgroup of a group in the class $\{1\}\mathcal{P}$ with derived length n . \diamond

Corollary 4.8 *If G is solvable in $PL_o(I)$ of derived length n , then G embeds in G_n .*

pf:

G embeds in $S(G)$, which also has derived length n . $S(G)$ is isomorphic to a group in $\{1\}\mathcal{P}$ with derived length n by Lemma 4.7. Now, by Lemma 2.2, $S(G)$ embeds in G_n . \diamond

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